Scientific Journal of Faculty of Education, Misurata University-Libya, Vol. 1, No. 10, Mar. 2018

Published on Web 01/3/2018

المجلة العلمية لكلية التربية، جامعة مصراتة، ليبيا، المجلد الأول ـ العدد العاشر، مارس 2018م

# EXTENSION OF POSITIVE LINEAR OPERATORS ON RIESZ SPACE

**MR. MANAL ATTAHER AZZAIDANI**

## Abstract

This research considers the extension of special functions called Positive Linear Operators on Riesz space, the positive linear operator whose domain is a Riesz subspace extends to a positive linear operator whose domain is all Riesz space if and only if it is dominated by a monotone sub linear function, and taking in consideration important theorem concerns extendable positive operator whose domain is an ideal always has a smallest extension. And .clarifies an extreme points of the convex set have been characterized.

**Keywords:** extension, Dedekind complete space, linear operator, Riesz space, sublinear function.

## Introduction

In this research we consider the extension of special functions called Positive Linear Operators on Riesz space, and important theorem concerns an extension properties of positive operators. The first result of this is a positive linear operator whose domain is a Riesz subspace extends to a positive linear operator whose domain is all Riesz space if and only if it is dominated by a monotone sub linear function. Finally, it was clarified that an extendable positive linear operator whose domain is an ideal always smallest extension.

## Definition1

A Riesz space (or a vector lattice) is an ordered vector space $E$ with the additional property that for each pair of elements $x, y \in E$ the supremum and infimum of the set $\{x, y\}$ both exist in $E$.

## Definition2

A Riesz space is called Dedekind complete whenever every nonempty subset that is bounded from above has supremum (or, equivalently, whenever every nonempty subset bounded from below has an infimum).

**Definition3**

A function $T : E \longrightarrow F$ between two vector spaces means **linear operator** that it satisfies the following axiom:

Holds for all $x, y \in E$ and all $\alpha, \beta \in \Re$. $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

**Definition4**

A function $P : G \longrightarrow F$ , where $G$ is a (real) vector space and $F$ is a Riesz space, is called **sub linear** whenever

(a) $p(x+y) \leq p(x) + p(y)$ Holds for all $x, y \in G$; and

(b) $p(\lambda x) = \lambda p(x)$ Holds for all $x \in G$ and all $0 \leq \lambda \in R$.

**Definition5**

The **extension** of $T : X \longrightarrow Y$ to a set $X \subseteq \hat{X}$ is an operator $\hat{T} : \hat{X} \longrightarrow Y$ Such that $\hat{T}(x) = T(x)$ for all $x \in X$ .

**Definition6**

A function $f : E \longrightarrow F$ between two Riesz spaces is called monotone whenever $x \leq y$ in $E$ implies $f(x) \leq f(y)$ in $F$ .

The starting point in the theory of positive operators is a fundamental extension theorem of L. V. Kantorovic. The importance of the result lies in the fact that in order for operator $T : E^+ \longrightarrow F^+$ to be the restriction of a (unique) positive operator from $E$ into $F$ , it is necessary and sufficient for $T$ to be additive on $E^+$. The details follow.

**Theorem 1** (Kantorovic). If $T : E^+ \longrightarrow F^+$ is additive ( $T(x+y) = T(x) + T(y)$ holds for all $x, y \in E^+$ ), then $T$ extends uniquely to a positive operator from $E$ into $F$ . Moreover, the unique extension ( denoted by $T$ again ) is given by

$$T(x) = T(x^+) - T(x^-)$$

For each $x \in E$ .

**Proof.** Clearly, if the operator $S : E \longrightarrow F$ is an extension of $T$ , then $S$ must be a positive operator and $S(x) = T(x^+) - T(x^-)$ must hold for each $x \in E$ .

That is, $T$ has at most one extension, and if it does have an extension, say $S$, then $S(x)=T(x^+)-T(x^-)$ must hold for each $x\in E$. Thus, what remains to be shown is that $S(x)=T(x^+)-T(x^-)$ defines an operator from $E$ into $F$.
The additivity of $S$ is established frist. To do this, we need the following property: If $x=y-z$ with $y,z\in E^+$, then $S(x)=T(y)-T(z)$ holds. ( Indeed, from $x=x^+-x^-=y-z$ it follows that $x^++z=y+x^-$, and so by the additivity of $T:E^+\longrightarrow F^+$ we see that

$$T(x^+)+T(z)=T(x^++z)=T(y+x^-)=T(y)+T(x^-).$$

Thus, $S(x)=T(x^+)-T(x^-)=T(y)-T(z)$ holds. Therefore, if $u,v\in E$, then

$$S(u+v)=S([u^++v^+]-[u^-+v^-])=T(u^++v^+)-T(u^-+v^-)$$
$$=[T(u^+)-T(u^-)]+[T(v^+)-T(v^-)]=S(u)+S(v).$$

For the homogeneity of $S$ we need the property: If $0\le y\le x$ holds in $E$, then $T(y)\le T(x)$ in $F$. (Indeed, from $0\le y\le x$ it follows that
$T(y)\le T(y)+T(x-y)=T(y+[x-y])=T(x)$.)
Now let $x\in E^+$ and $\lambda\ge 0$. Pick two sequences of rational numbers $\{r_n\}$ and $\{t_n\}$ with $0\le r_n\uparrow\lambda$ and $0\le t_n\uparrow\lambda$. Using the additivity of $T$ on $E^+$, we see that

$$r_nT(x)=T(r_nx)\le T(\lambda x)\le T(t_nx)=t_nT(x),$$

And from this it follows that $T(\lambda x)=\lambda T(x)$. Finally, let $x\in E$ and $\lambda\in\Re$.
If $\lambda\ge 0$, then

$$S(\lambda x)=T(\lambda x^+)-T(\lambda x^-)=\lambda T(x^+)-\lambda T(x^-)=\lambda S(x),$$

And if $\lambda<0$, then

$$S(\lambda x)=-S([-\lambda]x)=-(-\lambda)S(x)=\lambda S(x)$$

The proof of the theorem is now complete.
The next result is the most general version of what is known as the Hahn Banach extension theorem. This theorem plays a fundamental role in modern analysis and without exception will be of great importance to us here.

**Theorem 2** (Hahn-Banach). Let $G$ be a (real) vector space, $F$ a Dedekind complete Riesz space, and let $P:G\longrightarrow F$ be a sublinear function. If $H$ is a vector subspace of $G$ and $S:H\longrightarrow F$ is an operator satisfying $S(x) \le P(x)$ for all $x \in H$ , then there exists some operator $T : G \longrightarrow F$ such that

1. $T = S$ on $H$ (i.e., $T$ is an extension of $S$ ); and
2. $T(x) \le P(x)$ holds for all $x \in G$ .

**Proof**. The critical step is to show that $S$ has an extension satisfying (2) on an arbitrary vector subspace generated by $H$ and one extra element. If this is done, then an application of Zorn's lemma guarantees an extension .of $S$ to all of $G$ with the desired properties

To this end, let $x \notin H$ , and let $v = \{y + \lambda x : y \in H, \lambda \in \Re\}$ . If $T : V \longrightarrow F$ is an extension of $S$ , then

$$T(y + \lambda x) = S(y) + \lambda T(y)$$

must hold for all $y \in H$ and $\lambda \in \Re$. Put $z = T(x)$ . To complete the proof, we have to show the existence of some $z$ such that

$$S(y) + \lambda z \le p(y + \lambda x) \qquad (*)$$

holds for all $y \in H$ and $\lambda \in \Re$. For $\lambda > 0$ , $(*)$ is equivalent to

$$S(y) + z \le p(y - x)$$

for all $y \in H$ , while for $\lambda < 0$, $(*)$ is equivalent to

$$S(y) - z \le p(y - x)$$

for all $y \in H$ . The last two inequalities certainly will be satisfied for a choice of $z$ for which

$$S(y) - p(y - x) \le z \le P(u + x) - S(u) \qquad (**)$$

.holds for all $y, u \in H$ .

Finally, to see that there exists some $z \in F$ satisfying $(**)$, start by observing that for each $y, u \in H$ we have

$$S(y) + S(u) = S(y + u) \le p(y + u) = P(y - x + (u + x))$$
$$\le p(y - x) + P(u + x)$$

and so

$$S(y) - p(y - x) \le P(u + x) - S(u)$$

holds for all $y, u \in H$. This inequality coupled with the Dedekind completeness of $F$ guarantees that both

$$s = \sup\{S(y) - p(y - x) : y \in H\}$$

and

$$t = \inf\{P(u + x) - S(u) : u \in H\}$$

exist in $F$, and they satisfy $s \le t$. Now any $z \in F$ satisfying $s \le z \le t$ (for instance, $z = s$ ) satisfies $(**)$, and hence (*). The proof of theorem is now complete.

**Theorem 3** Let $T : E \longrightarrow F$ be a positive operator between two Riesz spaces with $F$ Dedekind complete. Assume also that $G$ is a Riesz subspace of E and that $S : G \longrightarrow F$ is an operator satisfying $0 \le S(x) \le T(x)$ for all $x \in G^+$. Then $S$ can be extended to a positive operator from $E$ into $F$ such that $0 \le S(x) \le T(x)$ for all $x \in E$

**Proof**. Let $P : E \longrightarrow F$ be defined by $p(x) = T(x^+)$, and note that $P$ is sublinear and that $S(x) \le p(x)$ holds for all $x \in G$. By Theorem 2 there exists an extension of $S$ to all of $E$ (which we denote by $S$ again) satisfying for all $x \in E$. Hence, if $x \in E^+$, then

$$-S(x) = S(-x) \le p(-x) = T\big((-x)^+\big) = T(0) = 0$$

And so $0 \le S(x) \le p(x) = T(x)$ holds, as desired.

The rest of the research is devoted to extension properties of positive operators. The first result of this kind tells us that a positive linear operator whose domain is a Riesz subspace extends to a positive linear operator whose domain is all Riesz space if and only if it is dominated by a monotone sub linear function.

**Theorem 4** Let $E$ and $F$ be Riesz spaces with $F$ Dedekind complete. If $G$ is a Riesz subspace of $E$ and $T : G \longrightarrow F$ is a positive operator, then the following statements are equivalent:

1. $T$ Extends to a positive operator (from $E$ into $F$ ).
2. $T$ Extends to an order bounded operator (from $E$ into $F$ ).
3. There exists a monotone sub linear function $P : E \longrightarrow F$ satisfying $T(x) \le P(x)$ for all $x \in G$

**Proof**. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Let $S \in \Im_b(X,Y)$ satisfy $S(x) = T(x)$ for all $x \in G$. Then $P : E \longrightarrow F$ defined by $p(x) = |S|(x^+)$ is monotone sub linear and satisfies

$$T(x) \leq T(x^+) = S(x^+) \leq |S|(x^+) = p(x)$$

For all $x \in G$.

(3) $\Rightarrow$ (1) Let $P : E \longrightarrow F$ be a monotone sublinear function satisfying $T(x) \leq P(x)$ for all $x \in G$ Then $q(x) = p(x^+)$ defines a sublinear function from $E$ into $F$ such that

$$T(x) \leq T(x^+) \leq p(x^+) = q(x)$$

Holds for all $x \in G$. Thus, by the Hahn-Banach theorem there exists an extension $R \in \Im(E,F)$ of $T$ satisfying $R(x) \leq q(x)$ for all $x \in E$. In particular, if $x \in E^+$, then the relation

$$-R(x) = R(-x) \leq q(-x) = p\left((-x)^+\right) = p(0) = 0$$

Implies that $0 \leq R(x)$ holds. That is, $R$ is a positive extension of $T$ to all of $E$, and the proof is finished.

**Definition7**

A subset $A$ of a Riesz space is called solid whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$.

**Definition8**

$A$ Solid vector subspace of a Riesz space is referred to as an ideal. From the identity $x \vee y = \frac{1}{2}\left(x + y + |x - y|\right)$, it follows immediately that every ideal is a Riesz subspace.

The next result deals with restrictions of positive operators to ideals.

**Theorem 5** If $T : E \longrightarrow F$ is a positive operator between two Riesz spaces with $F$ Dedekind complete, then for every ideal $A$ of $F$ the formula

$$T_A(x) = \sup\{T(y) : 0 \leq y \leq x \ ; \ y \in A, x \in E^+\}$$

Defines a positive operator from $E$ into $F$. Moreover, we have

1. $0 \leq T_A \leq T$
2. $T_A = T$ on $A$ and $T_A = 0$ on $A^d$; and
3. If $B$ is another ideal with $A \subseteq B$, then $T_A \leq T_B$ holds.

**Proof**. Note first that

$$T_A(x) = \sup\{T(x \wedge y) : y \in A^+\}$$

Holds for all $x \in E^+$. According to theorem1 it is enough to show that $T_A$ is additive on $x \in E^+$.

To this end, let $x, y \in E^+$. If $z \in A^+$, then inequality

$$(x + y) \wedge z \le x \wedge z + y \wedge z$$

Shows that

$$T((x + y) \wedge z) \le T(x \wedge z) + T(y \wedge z) \le T_A(x) + T_A(y)$$

And hence

$$T_A(x + y) \le T_A(x) + T_A(y)$$

On the other hand, the inequality

$$x \wedge u + y \wedge v \le (x + y) \wedge (u + v)$$

Implies that

$$T_A(x) + T_A(y) \le T_A(x + y)$$

Therefore, $T_A(x) + T_A(y) = T_A(x + y)$ holds, so that $T_A$ is additive on $E^+$.

Properties (1)-(3) are now an easy consequence of the formula defining $T_A$.

As we have mentioned before, if $G$ is a vector subspace of a Riesz space $E$, then it is standard to call an operator $T : G \to F$ positive whenever $0 \le x \in G$ implies $0 \le T(x) \in F$.

Consider a positive operator $T : G \to F$, where $G$ is a vector subspace of $E$. By $\xi(T)$ we shall denote the collection of all positive extensions of $T$ to all of $E$. That is,

$$\xi(T) = \{S \in \Im(E, F) : S \ge 0 \text{ and } S = T \text{ on } G\}.$$

An extendable positive operator whose domain is an ideal always has a smallest extension.

**Theorem6** Let $E$ and $F$ be two Riesz spaces with $F$ Dedekind complete, let $A$ be an ideal of $E$, and let $T : A \to F$ be a positive operator. If $\xi(T) \ne \varnothing$, then $T$ has a smallest extension. Moreover, if in this case $S = \min \xi(T)$, then

$$S(x) = \sup\{T(y) : y \in A, 0 \le y \le x\}$$

Holds for all $x \in E^+$ .

**Proof**. Since $T$ has (at least) one positive extension, the formula

$$T_A(x) = \sup\{T(y) : y \in A, 0 \le y \le x, x \in E^+\},$$

defines a positive operator from $E$ into $F$ satisfying $T_A = T$ on $A$, and so $T_A \in \xi(T)$. (See the proof of Theorem 4)

Now if $S \in \xi(T)$, then $S = T$ holds on $A$, and hence $T_A = S_A \leq S$ .

Therefore, $T_A = \min \xi(T)$ holds, as desired.

For a positive operator $T : E \longrightarrow F$ with $F$ Dedekind complete, the preceding theorem implies that for each ideal $A$ of $E$ the positive operator $T_A$ is the smallest extension of the restriction of $T$ to $A$.

Among the important points of a convex set are its extreme points.

**Definition9**

Recall that an element $e$ of a convex set $C$ is said to be an **extreme point** of $C$ whenever the expression $e = \lambda x + (1-\lambda)y$, with $x, y \in C$ and $0 < \lambda < 1$, implies $x = y = e$.

The extreme points of the convex set $\xi(T)$ have been characterized by Z. Lipecki, D. Plachky, and W. Thomsen as follows.

**Theorem 7** (Lipecki - Plachky – Thomsen). Let $E$ and $F$ be two Riesz spaces with $F$ Dedekind complete. If $G$ is a vector subspace of $E$ and $T : G \rightarrow F$ is a positive operator, then for an operator $S \in \xi(T)$ the following statements are equivalent:

1. $S$ is an extreme point of $\xi(T)$
2. $F$ or each $x \in E$ we have $\inf\{S(|x-y|) : y \in G\} = 0$

**Proof:** (1) $\Rightarrow$(2) Define $p : E \rightarrow F$ by

$$p(x) = \inf\{S(|x-y|) : y \in G\}$$

For each $x \in E$ .Clearly, $p$ is a sublinear mapping that satisfies $0 \leq p(x) = p(-x) \leq S(|x|)$ for all $x \in E$, and also $p(y) = 0$ for each $y \in G$.

Next, we claim that $p(y) = 0$ holds for all $x \in E$. To see this, assume by way of contradiction that $p(x) > 0$ holds for some $x \in E$. Define the operator $R : \{\lambda x : \lambda \in R\} \rightarrow F$ by $R(\lambda x) = \lambda p(x)$, and note that $R(\lambda x) \leq p(\lambda x)$ holds. By the Hahn-Banach theorem, $R$ has an extension to all of $E$ (which we shall denote by $R$ again) such that $R(z) \leq p(z)$ holds for all $z \in E$; clear, $R \neq 0$. It easily follows that $|R(z)| \leq p(z)$ holds, and so $R(y) = 0$ for all $y \in G$. Since for each $z \geq 0$ we have

$$R(z) \le p(z) \le S(|z|) = S(z)$$

And

$$-R(z) = R(-z) \le p(-z) \le S(|-z|) = S(z),$$

It is easy to see that $S - R \ge 0$ and $S + R \ge 0$ both hold. Thus, $S - R$ and $S + R$ both belong to $\xi(T)$. Now the relation

$$S = \frac{1}{2}(S - R) + \frac{1}{2}(S + R)$$

Coupled with $S - R \ne S$ and $S + R \ne S$, shows that $S$ is not an extreme point of $\xi(T)$, a contradiction. Thus, $p(x) = 0$ holds for each $x \in E$, as desired.

(2) $\Rightarrow$ (1) Let $S$ satisfy (2), and let $S = \lambda Q + (1-\lambda)R$ with $Q, R \in \xi(T)$ and $0 < \lambda < 1$. Then for each $x, y \in E$ we have

$$|Q(x) - Q(y)| \le Q|x - y| = \left(\frac{1}{\lambda}S - \frac{1-\lambda}{\lambda}R\right)|x - y| \le \frac{1}{\lambda}S|x - y|.$$

In particular, if $x \in E$ and $y \in G$, then it follows from $S(y) = Q(y) = T(y)$ that

$$|S(x) - Q(x)| \le |S(x) - S(y)| + |Q(y) - Q(x)| \le \left(1 + \frac{1}{\lambda}\right)S|x - y|.$$

Taking into account our hypothesis, the last inequality yields $S(x) = Q(x)$ for each $x \in E$, and this shows that $S$ is an extreme point of $\xi(T)$. The proof of the theorem

**References**


1. Carl L Devito, Functional analysis and Linear operator theory, Addisson Wesley publishing, Inc. 1990 .
2. Charalambos D. Aliprntis, Positive Operators, Acaddemic press, Inc. 1985.
3. 3 John B. Conway, A course in functional analysis, second edition, Springer-verlag, NewYork, Inc. 1990.
4. Klaus Donner, Extension of positive operators and Korovkin theorems, Springer-verlag, Berlin Heidelberg, NewYork, Inc. 1990.
5. Kreyszing E., Introductory functional analysis with applications, John Wiley & sons, Inc. 1978 .
6. L. V. KANTOROVIC, Linear operators in semi-ordered spaces, Mat. Sb. (N. S.) , 1940.